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I.Sc - Class - XI Date - 09-04-2020

Complex Numbers and Quadratic Eqns.

Miscellaneous Exercise.

Q.17. If α and β are different complex numbers with $|\beta|=1$, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$

Soln $\because$ $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\frac{(\beta-\alpha)\bar{\beta}}{(1-\bar{\alpha}\beta)\bar{\beta}}\right| = \left|\frac{(\beta-\alpha)\bar{\beta}}{\bar{\beta}-\bar{\alpha}\beta\bar{\beta}}\right|$

$= \left|\frac{(\beta-\alpha)\bar{\beta}}{\bar{\beta}-\bar{\alpha}}\right|$ $[\because \beta\bar{\beta}=|\beta|^2=|1|^2=1]$

$= \frac{|(\beta-\alpha)\bar{\beta}|}{|\bar{\beta}-\bar{\alpha}|}$ $\left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$

$= \frac{|\beta-\alpha||\bar{\beta}|}{|\overline{\beta-\alpha}|}$ $[\because |z_1 z_2| = |z_1||z_2| \text{ and } \bar{z}_1-\bar{z}_2 = \overline{z_1-z_2}]$

$= \frac{\cancel{|\beta-\alpha|}|\beta|}{\cancel{|\beta-\alpha|}}$ $[\because |\bar{z}|=|z|]$

$= |\beta| = 1.$

Q.18. Find the number of non-zero integral solutions of the eqn $|1-i|^x = 2^x$.

Soln: $\because |1-i|^x = 2^x \Rightarrow \sqrt{1^2+(-1)^2}^{\,x} = 2^x \Rightarrow \sqrt{(1+1)}^{\,x} = 2^x$

$\Rightarrow \sqrt{2}^{\,x} = 2^x \Rightarrow 2^{\frac{x}{2}} = 2^x \Rightarrow \frac{x}{2} = x \Rightarrow 2x-x=0$
$\Rightarrow x=0.$

But we need the non-zero solutions.

Hence, the number of solutions is zero.

Q.20. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m.

Soln: $\because \left(\frac{1+i}{1-i}\right)^m = 1$

$$\Rightarrow \left[\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right]^m = 1$$

$$\Rightarrow \left[\frac{(1+i)^2}{1-i^2}\right]^m = 1$$

$$\Rightarrow \left[\frac{1+2i+i^2}{1+1}\right]^m = 1 \quad [\because (1+i)^2 = 1^2+2i+i^2 \text{ and } i^2=-1]$$

$$= \cancel{1}+2i-\cancel{1} \text{ and } i^2=-1$$
$$= 2i$$

$$\Rightarrow \left[\frac{\cancel{1}+2i-\cancel{1}}{2}\right]^m = 1$$

$$\Rightarrow \left[\frac{\cancel{2}i}{\cancel{2}}\right]^m = 1$$

$$\Rightarrow i^m = 1$$

$$\Rightarrow (\sqrt{-1})^m = 1 \quad [\because i^2=-1 \Rightarrow i=\sqrt{-1}]$$

$$\Rightarrow (-1)^{\frac{m}{2}} = 1$$

$$\Rightarrow (-1)^{\frac{m}{2}} = (-1)^2$$

$$\Rightarrow \frac{m}{2} = 2$$

$$\Rightarrow m = 4$$

Hence, the least positive integral value of m is 4

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B.Sc. part-I (H). paper II. 09-04-2020.
Analytical Geometry of two dimensions

Q.1. Find the co-ordinates of the radical centre of the three circles $x^2+y^2+4x+7=0$, $2x^2+2y^2+3x+5y+9=0$ and $x^2+y^2+y=0$.
Hence, find the equation of a circle cutting these circles orthogonally.

Soln:- The radical axis of the first and the last circles is obtained by subtracting their equations from each other,
i.e. $x^2+y^2+4x+7-(x^2+y^2+y)=0 \Rightarrow 4x-y+7=0$ —— (I)

The radical axis of the last two circles is
$$2x^2+2y^2+3x+5y+9-2(x^2+y^2+y)=0$$
$$\Rightarrow 3x+3y+9=0 \Rightarrow x+y+3=0 \text{ —— II}$$
Solving I and II by cross-multiplication, we get
$$4x-y+7=0$$
$$x+y+3=0$$
$$\frac{x}{-3-7}=\frac{y}{7-12}=\frac{1}{4+1} \Rightarrow \frac{x}{-10}=\frac{y}{-5}=\frac{1}{5} \Rightarrow x=-2 \text{ and } y=-1$$
Hence, the radical centre is $(-2,-1)$.
The radical centre $P(-2,-1)$ will be the centre of the required circle cutting the three given circles orthogonally.
From P draw a tangent PT to any of the given circles, say first, Then $PT=\sqrt{(-2)^2+(-1)^2+4(-2)+7}=\sqrt{4+1-8+7}=\sqrt{4}=2$
This is the radius of the required circle.
Hence, the required equation of the circle is
$$(x+2)^2+(y+1)^2=(2)^2 \quad [\text{from the formula } (x-h)^2+(y-k)^2=a^2]$$
$$\Rightarrow x^2+4x+4+y^2+2y+1=4$$
$$\Rightarrow x^2+y^2+4x+2y+1=0$$
which is the required eqn of the circle

Q.2. Find the general ~~solution~~ equation to the system of co-axal circle containing the circle $x^2+y^2=9$ and the limiting point $(2,3)$. Find also the other limiting point.

Soln: The equation of the point-circle at $(2,3)$ is

$$(x-2)^2+(y-3)^2=0$$
$$\Rightarrow x^2-4x+4+y^2-6y+9=0$$
$$\Rightarrow x^2+y^2-4x-6y+13=0 \longrightarrow \mathrm{I}$$

The general equation of the system of co-axal circles containing the given circle $x^2+y^2-9=0$ and the point circle (I) is

$$x^2+y^2-4x-6y+13+\lambda(x^2+y^2-9)=0$$
$$\Rightarrow (1+\lambda)(x^2+y^2)-4x-6y+(13-9\lambda)=0$$
$$\Rightarrow x^2+y^2-2\cdot\frac{2}{1+\lambda}x-2\frac{3}{1+\lambda}y+\frac{13-9\lambda}{1+\lambda}=0$$

Its centre is $\left(\frac{2}{1+\lambda},\frac{3}{1+\lambda}\right) \longrightarrow \mathrm{II}$

and radius is $\sqrt{\frac{4}{(1+\lambda)^2}+\frac{9}{(1+\lambda)^2}-\frac{13-9\lambda}{1+\lambda}}$.

For limiting points of the system, the radius must be zero.

$$\therefore \frac{4}{(1+\lambda)^2}+\frac{9}{(1+\lambda)^2}-\frac{13-9\lambda}{1+\lambda}=0$$
$$\Rightarrow 13-(13-9\lambda)(1+\lambda)=0 \Rightarrow 13-(13+4\lambda-9\lambda^2)=0$$
$$\Rightarrow \lambda(9\lambda-4)=0 \Rightarrow \lambda=0 \text{ or } \lambda=\frac{4}{9}.$$

When $\lambda=0$, from II, the limiting point is $(2,3)$, which is the given limiting point.

When $\lambda=\frac{4}{9}$, from II, the required limiting point is

$$\left(\frac{18}{13},\frac{27}{13}\right)$$

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B.Sc.-part-II(H), paper-III
Sequence and Series. Date-09-04-2020.

Q.1. State and prove Raabe's test.

Soln: Statement:- Let us suppose that $u_n > 0$ and that $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$

Then the series $\sum u_n$ is convergent if $l > 1$ and divergent if $l < 1$.

Proof: Let us compare the given series $\sum u_n$ with the auxiliary series

$$\sum v_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

Whose nth term $v_n = \frac{1}{n^p}$

We know that the series $\sum v_n$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

Now, $\frac{v_n}{v_{n+1}} = \frac{1}{n^p} \Big/ \frac{1}{(n+1)^p}$

$$= \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n}\right)^p = \left(1 + \frac{1}{n}\right)^p = 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \cdots$$

$$= 1 + \frac{p}{n} + \text{terms of the order } \frac{1}{n^2}$$

$$= 1 + \frac{p}{n} + o\left(\frac{1}{n^2}\right)$$

Here, there are two cases's arise

Case I. Let us suppose that $\sum v_n$ is convergent and hence $p > 1$

Then by the comparison test, $\sum u_n$ is convergent if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

i.e. if $\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + o\left(\frac{1}{n^2}\right)$

i.e. if $\frac{u_n}{u_{n+1}} - 1 > \frac{p}{n} + O\left(\frac{1}{n^2}\right)$

i.e. if $n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + O\left(\frac{1}{n}\right)$

i.e. if $Lt\, n\left(\frac{u_n}{u_{n+1}} - 1\right) > p(>1)$

i.e. if $Lt\, n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1$.

Case II Let us suppose that ΣV_n is divergent and hence $p<1$. Then Σu_n is divergent

if $\frac{u_n}{u_{n+1}} < \frac{V_n}{V_{n+1}}$

i.e. if $Lt\, n\left(\frac{u_n}{u_{n+1}} - 1\right) < p$

i.e. if $Lt\, n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1$

Proved

Q.2. Test the convergency of the series $1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{1.2.3} + \ldots$

Soln: We have, $u_n = \frac{a(a+1)(a+2)\cdots(a+n-1)}{1.2.3\cdots n}$ —— I

$u_{n+1} = \frac{a(a+1)(a+2)\cdots(a+n-1)(a+n)}{1.2.3\cdots n(n+1)}$ —— II

Dividing I by II, we get:

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{a+n} = \frac{n+1}{n+a}$$

$$\therefore \underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = 1$$

Hence, ratio test fails and we apply the Raabe's Test.

Now, $\frac{u_n}{u_{n+1}} - 1 = \frac{n+1}{n+a} - 1 = \frac{\cancel{n}+1-\cancel{n}-a}{n+a} = \frac{1-a}{n+a}$

$\therefore n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n(1-a)}{n+a}$ $\therefore \underset{n\to\infty}{Lt}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = Lt \frac{n}{n+a}(1-a) = 1-a$

Therefore, if $1-a>1$ i.e. $a<0$, the series is convergent

and if $1-a<1$ i.e. $a>0$, the series is divergent.

But if $1-a=1$ i.e. $a=0$, the series becomes

$1+0+0+0+0+\cdots$ which is clearly convergent.

Thus the series is convergent if $a \le 0$ and is divergent if $a>0$.

Which is the required Test.